Math 151 2015XB – Week 4 Day Tues

Review

Recall ~~[illegible]~~ simple and compounded interest equations

$$A = P(1 + rt)$$ simple ~~[illegible]~~ interest

principal

interest rate

A linearly related to t

$$A = P(1+r)^t$$ compound interest compounded annually

$$A = P(1 + r/m)^{mt}$$ compounded m times per year

$$= P(1+i)^n$$ compounded n times total

$n = mt$

i is interest rate per compounding period $= r/m$

we'll use this form today

$$A = Pe^{rt}$$ compounded continuously

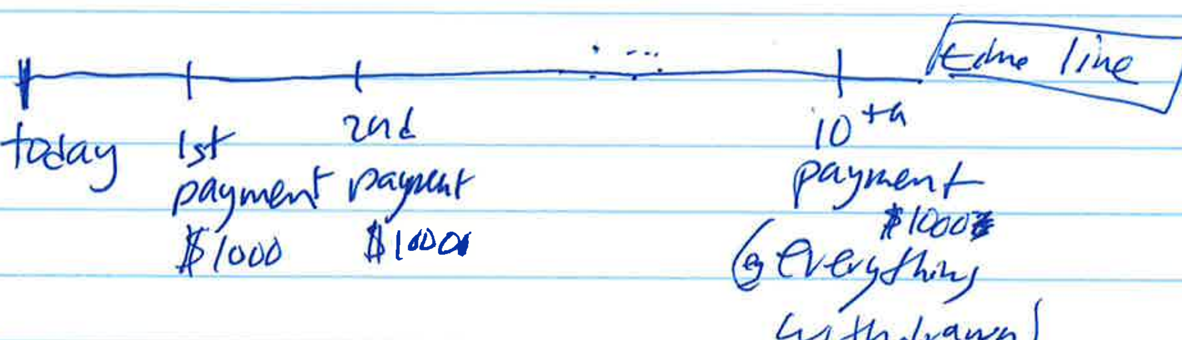
Section 3.3 - Future Value of an Annuity; Sinking Funds

An <u>annuity</u> is any sequence of equal periodic payments.

An <u>ordinary annuity</u> is one in which payments are made at the <u>end</u> of each interval

All annuities we consider are ordinary.

The <u>future value</u> of an annuity is the sum of all payments made plus interest earned up to a date in the future



How much is it worth when you withdraw everything (depends on interest rate, say 10%)

1st payment — $1000 + 9 years compounded interest on $1000.00

2nd payment — + $1000 + 8 years compounded interest on $1000.00

⋮

10th payment — + $1000 + no interest

Total amount that it is worth is called S

$1.1 = (1 + r)$ ← interest rate

$$S = 1000 + 1000\cdot(1.1) + (1000)(1.1)^2 + \dots + 1000(1.1)^9$$

↑ 10th payment, ↑ 9th payment, ↑ 8th payment, ↑ 1st payment

There is a trick to finding S. You could plug this all into your calculator But that would be tedious

If you are doing annuities with payments every month over 30 years you would have 360 terms in you sum

$$12 \cdot 30 = \boxed{360}$$

Uck!

There is an easier way to do it

$$1.1S = 1000\cdot 1.1 + 1000\cdot 1.1^2 + \dots + 1000\cdot 1.1^9 + 1000\cdot 1.1^{10}$$

$$- \quad S = 1000 + 1000\cdot 1.1 + 1000\cdot 1.1^2 + \dots + 1000\cdot 1.1^9$$

$$1.1S - S = 1.1^{10}\cdot 1000 - 1000$$

$$S(1.1-1) = 1000(1.1^{10} - 1)$$

$$S = 1000\cdot\left(\frac{1.1^{10}-1}{1.1-1}\right) = 1000\,\frac{1.1^{10}-1}{0.1}$$

15,937.42

\$100 payments every 6 month for 3 years
6% companded semiannually

0 1year 2year 3year
X 1 2 3 4 5 6

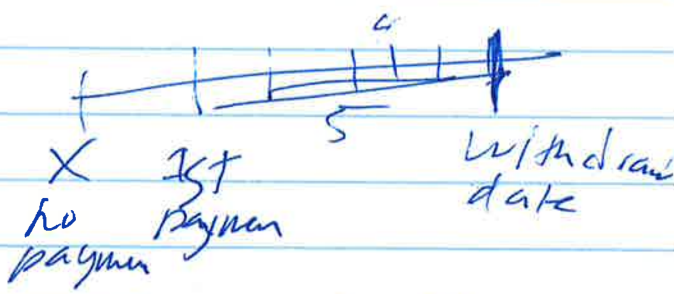
$n = 3\text{ years} \times 2 \text{ payments/year} = 6$ payments
$n = t \cdot m$
$i = r/m = 6\%/2 = 3\% = 0.03$

$A = \boxed{P_{MT}}(1+i)^n$ — Use PMT in equation for payment not P for principal each payments is like a principal but there are many of them so we use PMT to refer to payments

FV of 1st payment

FV $PMT(1+i)^5$

FV of 2nd payment $PMT(1+i)^4$
last payment PMT
All payments



$S = PMT + PMT(1+i)^1 + PMT(1+i)^2 + PMT(1+i)^3 + PMT(1+i)^4 + PMT(1+i)^5 + \cancel{PMT(1+i)}$

count 6 of them

Now trick

$$S \times (1+i) =$$

$$\left(PMT + PMT(1+i) + PMT(1+i)^2 + PMT(1+i)^3 + PMT(1+i)^4 + PMT(1+i)^5\right)(1+i)$$

$$= PMT(1+i) + PMT(1+i)^2 + PMT(1+i)^3 + PMT(1+i)^4 + PMT(1+i)^5 + PMT(1+i)^6$$

$$S(1+i) - S$$

$$= PMT(1+i)^6 - PMT$$

$$S((1+i)-1) = PMT((1+i)^6 - 1)$$

$$S = PMT\left(\frac{(1+i)^6 - 1}{(1+i)-1}\right)$$

$$= PMT\,\frac{(1+i)^6 - 1}{i}$$

This is for $n=6$. If we leave n as unknown get

$$FV = PMT \cdot \frac{(1+i)^n - 1}{i}$$

FV = Future Value (amount)
PMT = periodic payment
i = interest rate per period
n = number of payments (periods)

Eg 1: What is value of annuity at end of 20 years if \$2000 deposited each year into an account earning 8.5% compounded annually?

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 2000 \cdot \frac{(1+.085)^{20} - 1}{0.085}$$

$$= 96754.03$$

How much is interest?

$96754.03 - 2000 \cdot 20$

\$ 56,754.03

Matched 1: What is the value of an annuity at the end of 10 years if $1000 is deposited every 6 months in an account earning 8% compounded semiannually?

How much of this value is interest?

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

$$= 1000\,\frac{(1.04)^{20} - 1}{.04}$$

$= 29,778.08$ total

9788.08 interest.

Sinking Funds

Remember Equation

$$FV = PMT \frac{(1+i)^n - 1}{i}$$

The idea of a sinking fund is you solve this equation for PMT

You want to save \$100,000 by the time your kid goes to college. ~~that~~ ~~So~~ You know how much time that will be $n = 18$ years or $18 \cdot 12$ months if you contribute monthly. You have an investment that ~~guar~~ guarantees a perperiod interest rate i, You want to know PMT

Solve for PMT — Do it

$$PMT = FV \cdot \frac{i}{(1+i)^n - 1}$$

#52: A company estimates that it will have to replace a piece of equipment at a cost of $800,000 in 5 years. To have this money available in 5 years a sinking fund is established by making equal monthly payments into an account paying 6.6% compounded monthly

How much should each payment be.

$$PMT = FV \frac{i}{(1+i)^n - 1}$$

$$= 800{,}000 \cdot \frac{\frac{6.6}{12}}{\left(1+\frac{6.6}{12}\right)^{5 \cdot 12} - 1}$$

$$= 800{,}000 \cdot \frac{0.0055}{(1.0055)^{60} - 1}$$

$11,290.41 per month

$$FV = PMT\frac{(i+1)^n-1}{i}$$

Solve for n

$$\frac{i\,FV}{PMT}+1=(i+1)^n$$

$$\frac{\log\left(\frac{i\,FV}{PMT}+1\right)}{\log(i+1)}=n$$

$$FV = PMT\frac{(i+1)^n-1}{i}$$

Solve for i

$\boxed{\text{Can't do it! with algebra}}$

Need approximating software or graphical solutions.