Math 151 - 2015XB - Week 4 Mon (Pg 1)

§2.5 Exponential Functions

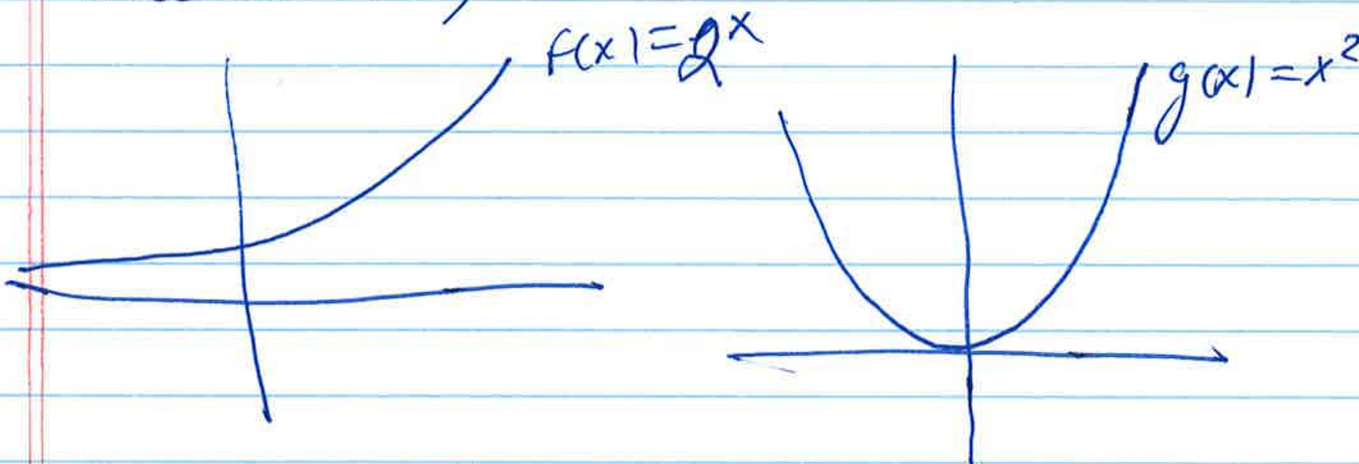
- growth of money through interest (compound)
- growth of populations
- radioactive decay

Note functions

$f(x)=2^x$ exponential function
$g(x)=x^2$ quadratic

For $f(x)=2^x$ the variable is in exponent
For $g(x)=x^2$ the variable is in the base

These are very different functions



There are different kinds of exponential functions:

$f(x)=b^x$ eg $f(x)=2^x$ $f(x)=4^x$
↑ for some constant b $f(x)=\left(\frac{1}{2}\right)^x$

Example from Appendix 1 Evaluate the following

(A) $4^{1/2}$ 2

$\sqrt{4}$ 2 $\sqrt{4} \neq \pm 2$

(B) $-4^{1/2}$ -2

$-\sqrt{4}$ -2

(C) $(-4)^{1/2}$ not real

$\sqrt{-4}$ not real

(D) $8^{1/3}$ 2

$\sqrt[3]{8}$ 2

(E) $(-8)^{1/3}$ -2

$\sqrt[3]{-8}$ -2

F $-8^{1/3}$ -2

$-\sqrt[3]{8}$ -2

Matched

(A) $16^{1/2}$ (B) $-\sqrt{16}$ (C) $\sqrt[3]{-27}$

(D) $(-9)^{1/2}$ (E) $(\sqrt[4]{81})^3$

Definition **Integer exponents**

1. For n a positive integer

$$a^n = a \cdot a \cdot a \cdots \cdot a$$

2. For $n = 0$

$$a^0 = 0$$

3. For n a negative integer

$$a^n = \frac{1}{a^{-n}}$$ ← positive integer

4. For all integers n

$$a^{-n} = \frac{1}{a^n}, \quad a^n = \frac{1}{a^{-n}}, \quad a \neq 0$$

Properties of exponents

1. $a^m a^n = a^{m+n}$
2. $(a^n)^m = a^{nm}$
3. $(ab)^m = a^m b^m$
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad b \neq 0$
5. $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}, \quad a \neq 0$

<u>Example</u>

(A) $(2x^3)(3x^5) = 2 \cdot 3x^{3+5} = 6x^8$

(B) $x^5 x^{-9} = x^{-4} = \frac{1}{x^4}$

(C) $\frac{x^5}{x^7} = x^{5-7} = x^{-2} = \frac{1}{x^2}$

(D) $\frac{x^{-3}}{y^{-4}} = \frac{y^4}{x^3}$

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(F) $\left(\frac{y^{-5}}{y^{-2}}\right)^{-2} = \frac{(y^{-5})^{-2}}{(y^{-2})^{-2}} = \frac{y^{10}}{y^4} = y^6$

(E) $(u^{-3}v^2)^{-2} = (u^{-3})^{-2}(v^2)^{-2}$

$= u^6 v^{-4} = \frac{u^6}{v^4}$

(G) $\frac{4m^{-3}n^{-5}}{6m^{-4}n^3} = \frac{2m^{-3-(-4)}}{3n^{3-(-5)}} = \frac{2m}{3n^8}$

(Matched)

(A) $(3y^4)(2y^3)$ | $= 6y^7$

(B) $m^2 m^{-6} =$ | $\frac{1}{m^4}$

(C) $(u^3v^{-2})^{-2}$ | $= u^{-6}v^4$ $= v^4/u^6$

(d) $\left(\frac{y^{-6}}{y^{-2}}\right)^{-1}$ $= \frac{y^6}{y^2} = y^4$

(E) $\frac{8x^2y^{-4}}{6x^{-5}y^2}$

$= \frac{8x^7}{6y^6} = \frac{4x^7}{3y^6}$

Eg2 Write $\dfrac{1-x}{x^{-1}-1}$ as a simple fraction with simple positive exponent

$$= \frac{x-x^2}{1-x} = \frac{x(1-x)}{(1-x)} = x \quad x \neq 1$$

Write $\dfrac{1+x^{-1}}{1-x^{-2}}$

$$= \frac{x^2(1+x^{-1})}{x^2(1-x^{-2})}$$

$$= \frac{x^2+x}{x^2-1} = \frac{x(1+x)}{(x+1)(x-1)} = \frac{x}{x-1}$$

Note $\sqrt{4}$ stands for single number
2 not ± 2. Do not confuse
$\sqrt{4}$ with the solutions to equation
$x^2 = 4$ written as $x = \pm\sqrt{4} = \pm 2$

Definition: If m and n are natural numbers with out prime factors, b is a real number and <u>b is nonnegative when n is even</u> then

$$b^{m/n} = \left(b^{1/n}\right)^m = \left(\sqrt[n]{b}\right)^m$$

$$= \left(b^m\right)^{1/n} = \sqrt[n]{b^m}$$

and

$$b^{-m/n} = \frac{1}{b^{m/n}}$$

CONCEPTUAL INSIGHT

All the properties for integer exponents listed in Theorem 1 in Section A.5 also hold for rational exponents, provided that b is nonnegative when n is even. This restriction on b is necessary to avoid nonreal results. For example,

$$(-4)^{3/2} = \sqrt{(-4)^3} = \sqrt{-64} \quad \text{Not a real number}$$

To avoid nonreal results, all variables in the remainder of this discussion represent positive real numbers.

EXAMPLE 2 From Rational Exponent Form to Radical Form and Vice Versa Change rational exponent form to radical form.

(A) $x^{1/7} = \sqrt[7]{x}$

(B) $(3u^2v^3)^{3/5} = \sqrt[5]{(3u^2v^3)^3}$ or $(\sqrt[5]{3u^2v^3})^3$ The first is usually preferred.

(C) $y^{-2/3} = \dfrac{1}{y^{2/3}} = \dfrac{1}{\sqrt[3]{y^2}}$ or $\sqrt[3]{y^{-2}}$ or $\sqrt[3]{\dfrac{1}{y^2}}$

Change radical form to rational exponent form.

(D) $\sqrt[5]{6} = 6^{1/5}$

(E) $-\sqrt[3]{x^2} = -x^{2/3}$

(F) $\sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$ Note that $(x^2 + y^2)^{1/2} \neq x + y$. Why?

Matched Problem 2 Convert to radical form.

(A) $u^{1/5}$ (B) $(6x^2y^5)^{2/9}$ (C) $(3xy)^{-3/5}$

Convert to rational exponent form.

(D) $\sqrt[4]{9u}$ (E) $-\sqrt[7]{(2x)^4}$ (F) $\sqrt[3]{x^3 + y^3}$

EXAMPLE 3 Working with Rational Exponents Simplify each and express answers using positive exponents only. If rational exponents appear in final answers, convert to radical form.

(A) $(3x^{1/3})(2x^{1/2}) = 6x^{1/3+1/2} = 6x^{5/6} = 6\sqrt[6]{x^5}$

(B) $(-8)^{5/3} = [(-8)^{1/3}]^5 = (-2)^5 = -32$

(C) $(2x^{1/3}y^{-2/3})^3 = 8xy^{-2} = \dfrac{8x}{y^2}$

(D) $\left(\dfrac{4x^{1/3}}{x^{1/2}}\right)^{1/2} = \dfrac{4^{1/2}x^{1/6}}{x^{1/4}} = \dfrac{2}{x^{1/4-1/6}} = \dfrac{2}{x^{1/12}} = \dfrac{2}{\sqrt[12]{x}}$

Matched Problem 3 Simplify each and express answers using positive exponents only. If rational exponents appear in final answers, convert to radical form.

(A) $9^{3/2}$ (B) $(-27)^{4/3}$ (C) $(5y^{1/4})(2y^{1/3})$ (D) $(2x^{-3/4}y^{1/4})^4$

(E) $\left(\dfrac{8x^{1/2}}{x^{2/3}}\right)^{1/3}$

EXAMPLE 4 Working with Rational Exponents Multiply, and express answers using positive exponents only.

(A) $3y^{2/3}(2y^{1/3} - y^2)$ (B) $(2u^{1/2} + v^{1/2})(u^{1/2} - 3v^{1/2})$

SOLUTION

(A) $3y^{2/3}(2y^{1/3} - y^2) = 6y^{2/3+1/3} - 3y^{2/3+2}$

$= 6y - 3y^{8/3}$

(B) $(2u^{1/2} + v^{1/2})(u^{1/2} - 3v^{1/2}) = 2u - 5u^{1/2}v^{1/2} - 3v$

Matched Problem 4 Multiply, and express answers using positive exponents only.

(A) $2c^{1/4}(5c^3 - c^{3/4})$ (B) $(7x^{1/2} - y^{1/2})(2x^{1/2} + 3y^{1/2})$

EXAMPLE 5 Working with Rational Exponents Write the following expression in the form $ax^p + bx^q$, where a and b are real numbers and p and q are rational numbers:

$$\frac{2\sqrt{x} - 3\sqrt[3]{x^2}}{2\sqrt[3]{x}}$$

SOLUTION

$$\frac{2\sqrt{x} - 3\sqrt[3]{x^2}}{2\sqrt[3]{x}} = \frac{2x^{1/2} - 3x^{2/3}}{2x^{1/3}}$$ Change to rational exponent form.

$$= \frac{2x^{1/2}}{2x^{1/3}} - \frac{3x^{2/3}}{2x^{1/3}}$$ Separate into two fractions.

$$= x^{1/6} - 1.5x^{1/3}$$

Matched Problem 5 Write the following expression in the form $ax^p + bx^q$, where a and b are real numbers and p and q are rational numbers:

$$\frac{5\sqrt[3]{x} - 4\sqrt{x}}{2\sqrt{x^3}}$$

Properties of Radicals

Changing or simplifying radical expressions is aided by several properties of radicals that follow directly from the properties of exponents considered earlier.

THEOREM 1 Properties of Radicals

If n is a natural number greater than or equal to 2, and if x and y are positive real numbers, then

1. $\sqrt[n]{x^n} = x$ $\qquad \sqrt[3]{x^3} = x$
2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\qquad \sqrt[5]{xy} = \sqrt[5]{x}\sqrt[5]{y}$
3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$ $\qquad \sqrt[4]{\dfrac{x}{y}} = \dfrac{\sqrt[4]{x}}{\sqrt[4]{y}}$

EXAMPLE 6 Applying Properties of Radicals Simplify using properties of radicals.

(A) $\sqrt[4]{(3x^4y^3)^4}$ (B) $\sqrt[4]{8}\sqrt[4]{2}$ (C) $\sqrt[3]{\dfrac{xy}{27}}$

SOLUTION

(A) $\sqrt[4]{(3x^4y^3)^4} = 3x^4y^3$ Property 1

(B) $\sqrt[4]{8}\sqrt[4]{2} = \sqrt[4]{16} = \sqrt[4]{2^4} = 2$ Properties 2 and 1

(C) $\sqrt[3]{\dfrac{xy}{27}} = \dfrac{\sqrt[3]{xy}}{\sqrt[3]{27}} = \dfrac{\sqrt[3]{xy}}{3}$ or $\dfrac{1}{3}\sqrt[3]{xy}$ Properties 3 and 1

Matched Problem 6 Simplify using properties of radicals.

(A) $\sqrt[7]{(x^3 + y^3)^7}$ (B) $\sqrt[3]{8y^3}$ (C) $\dfrac{\sqrt[3]{16x^4y}}{\sqrt[3]{2xy}}$

What is the best form for a radical expression? There are many answers, depending on what use we wish to make of the expression. In deriving certain formulas, it is sometimes useful to clear either a denominator or a numerator of radicals.

(Eg) Chang rational exponent to radical form

(A) $x^{1/7} = \sqrt[7]{x}$

(B) $(3u^2v^3)^{3/5} = \sqrt[5]{(3u^2v^3)^3}$

$= \sqrt[5]{27u^6v^9}$

(C) $y^{-2/3} = \frac{1}{y^{2/3}} = \frac{1}{\sqrt[3]{y^2}}$ or $\sqrt[3]{\frac{1}{y^2}}$

change radical to rational

(D) $\sqrt[5]{6} = 6^{1/5}$

(F) $\sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$ $\neq x+y$

(E) $-\sqrt[3]{x^2} = -x^{2/3}$

Matched 2

Convert to radical

(A) $u^{1/5}$ (B) $(6x^2y^5)^{2/9}$

$\sqrt[5]{u}$ $= \sqrt[9]{36x^4y^{10}}$

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(c) $(3xy)^{-3/5} = \sqrt[5]{\frac{1}{(3xy)^3}}$

Convert to rational exponent

(a) $\sqrt[4]{9u} = (9u)^{1/4}$

(b) $-\sqrt[7]{(2x)^4}$

$= -(2x)^{4/7}$

(c) $\sqrt[3]{x^3+y^3}$

$= (x^3+y^3)^{1/3}$

Eg 3 Simplify each and express answers using positive exponents only. Convert rational exponents to radicals.

(A) $(3x^{1/3})(2x^{1/2}) = 6x^{1/3+1/2} = 6x^{5/6}$

$= 6\sqrt[6]{x^5}$

(B) $-8^{5/3} = ((-8)^{1/3})^5 = (-2)^5 = -32$

(C) $(2x^{1/3}y^{-2/3})^3 = \frac{8x}{y^2}$

(D) $\left(\frac{4x^{1/3}}{x^{1/2}}\right)^{1/2} = \frac{4^{1/2}x^{1/6}}{x^{1/4}}$

$= \frac{2}{x^{1/4-1/6}} = \frac{2}{x^{3/12-2/12}}$

$= \frac{2}{x^{1/12}} = \frac{2}{\sqrt[12]{x}}$

Matched 3 Same instructions

(A) $9^{3/2} = 3^3 = 27$

(B) $(-27)^{4/3} = (-3)^4 = 81$

(C) $(5y^{1/4})(2y^{1/3}) =$

$10y^{1/4+1/3} = 10y^{\left(\frac{3}{12}+\frac{4}{12}\right)} = 10y^{7/12}$

$10\sqrt[12]{y^7}$

(D) $(2x^{-3/4}y^{1/4})^4$

$2x^{-3}y = \frac{2y}{x^3}$

(E) $\left(\frac{8x^{1/2}}{x^{2/3}}\right)^{1/3} = \frac{8x^{1/6}}{x^{2/9}}$

$8x^{1/6-2/9}$

$8x^{\left(3/18-4/18\right)}$

$\frac{8}{x^{1/18}} = \frac{8}{\sqrt[18]{x}}$

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Matched 4

(A) $2c^{1/4}(5c^{3} - c^{3/4})$

$10c^{13/4} - c^{3/16}$

(B) $(7x^{1/2} - y^{1/2})(2x^{1/2} + 3y^{1/2})$

$= 14x + 21x^{1/2}y^{1/2} - 2x^{1/2}y^{1/2} - 3y$

$14x - 19x^{1/2}y^{1/2} - 3y$

Matched 5

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$$\frac{5\sqrt[3]{x} - 4\sqrt{x}}{2\sqrt{x^3}}$$

$$= \frac{5x^{1/3} - 4x^{1/2}}{2 \cdot x^{3/2}}$$

$$= \frac{5}{2}x^{1/3 - 3/2} - \frac{4}{2}x^{1/2 - 3/2}$$

$$= \frac{5}{2}x^{+\frac{2}{6} - \frac{9}{6}}$$

$$\frac{5}{2}x^{-7/6} - \frac{4}{2}x^{-1/2}$$

$$\frac{5}{2}x^{-7/6} - 2x^{-1/2}$$

Method 6

A $x^3 + y^3$

(B) $2y$

(C) $$\frac{\sqrt[3]{2\cdot 2\cdot 2\cdot 2}\quad x^{4/3}\quad y^{1/3}}{\sqrt[3]{2}\quad x^{1/3}\quad y^{1/3}}$$

$2 \cdot x$

$2x$